Non-Oriented Heegaard Diagrams

Hossein Hatamian
(Taras Shevchenko National University of Kyiv 64/13, Volodymyrska Street, Kyiv, Ukraine)
E-mail: myowngait@gmail.com

Alexandr Prishlyak
(Taras Shevchenko National University of Kyiv, 64/13, Volodymyrska Street, Kyiv, Ukraine)
E-mail: prishlyak@yahoo.com

Let $(F; u_1, u_2; v_1, v_2)$ be a genus 2 Heegaard diagram of a closed 3-manifold M. Here F is a closed surface which decomposes M into two handlebodies H_1, H_2 of genus 2, u_1, u_2 are meridians of H_1, and v_1, v_2 are meridians of H_2.

Assume that all crossing points of the meridians are transversal and that the diagram is normalized (the latter means that among the regions into which the meridians split F here are no biangles). The total number of those crossing points is called the *Heegaard complexity* of the diagram.

Let us cut F along u_1, u_2. We obtain a sphere with four holes $D_1^{\pm}$, $D_2^{\pm}$ which are conveniently interpreted as distinguished disks on the sphere. The meridians v_1, v_2 will then be cut into arcs which join the holes. We agree to depict k parallel arcs as one arc marked by the number $k \geq 0$. We also take into account the orientations of the glued boundaries $\partial D_i^{\pm}$ by assigning $(+1)$ for orientable gluing and (-1) for non-orientable one. Therefore we get $(+1, +1)$, $(+1, -1)$, $(-1, +1)$ and $(-1, -1)$ for pairs of holes.

It is well known that the set of all genus 2 Heegaard diagrams can be decomposed into three types. Each such diagram can be determined by a 7-tuple (a, b, c, d, e, f, g), where a, b, c, d are the arcs joining the holes, e, f determine the gluing maps $\varphi_i \colon \partial D_i^- \to \partial D_i^+$, $i = 1, 2$, and g is defined to be 0, 1, 2 and 3 for $(+1, +1)$, $(+1, -1)$, $(-1, +1)$ and $(-1, -1)$ respectively.

In order to give exact descriptions of φ_1, we introduce topological symmetries $s_i \colon \partial D_i^- \to \partial D_i^+$ and topological rotations $r_i \colon \partial D_i^- \to \partial D_i^+$ by the following results:

(1) $s_i \colon \partial D_i^- \cap (v_1 \cup v_2) \to \partial D_i^+ \cap (v_1 \cup v_2)$ such that the endpoint of each b-arc (respectively, c-arc) is taken to the other endpoint of the same arc.

(2) r_i shifts each point of $D_i^+ \cap (v_1 \cup v_2)$ to the next point of $D_i^+ \cap (v_1 \cup v_2)$.

Now we define φ_1, φ_2 as follows: $\varphi_1 = r_1^e s_1$, and $\varphi_2 = r_2^f s_2$.

A Heegaard diagram $(F; u_1, u_2; v_1, v_2)$ of genus 2 has type I, II or III if it can be described as follows:

Type I

- Each a-arc joins D_1^+ and D_2^+ as well as D_1^- and D_2^-.
- Each b-arc joins D_1^+ and D_1^-.
- Each c-arc joins D_2^+ and D_2^-.
- Each d-arc joins D_1^+ and D_2^- as well as D_1^- and D_2^+.

Type II

- Each a-arc, b-arc and d-arc can be described in the same way as type I while each c-arc joins D_1^+ and D_1^-.

Type III

- Each a-arc, b-arc and c-arc can be described in the same way as type II while each d-arc is a loop with ends at D_1^+ and D_1^- embracing D_2^+ and D_2^- respectively.

Theorem 1. *If the Heegaard complexity of a non-orientable 3-manifold M is no more than 5, then M is homeomorphic to $L_{p,q} \# S^1 \widetilde{\times} S^2$.*

There exists a manifold of complexity 6 that is not homeomorphic to $L_{p,q}\#S^1\widetilde{\times}S^2$*. Its diagram can be represented by the 7-tuple* $(1,1,1,1,0,0,3)$.

References

[1] S.V. Matveev, *Algorithmic topology and classification of 3-manifolds*, volume 9, Springer, Berlin, 2007.